Polar Coordinate Overview

Polar coordinates are an essential tool for working with circles and spirals in 2D. We will need polar coordinates at various points throughout the rest of the quarter, but especially in chapter 15 when we are trying to find volumes above circular regions.

Introduction

You should all be very familiar with the *Cartesian coordinate method* for describing points

Cartesian method: (x, y)

1. Stand on the origin.
2. First, walk *x* units on the *x*-axis.
3. Then, walk *y* units parallel to the *y*-axis.

This is kind of like giving driving directions ("drive x blocks over, then y-blocks up").
This method works well when you are dealing with a problem/function in terms of *x*.

However, in some scenarios it is more convenient to give the location in terms of an ***angle*** and a ***radius***. For example, imagine you are firing a cannon; you need to know where to aim it and how powerful to shoot it. We call this the *Polar coordinate method*.

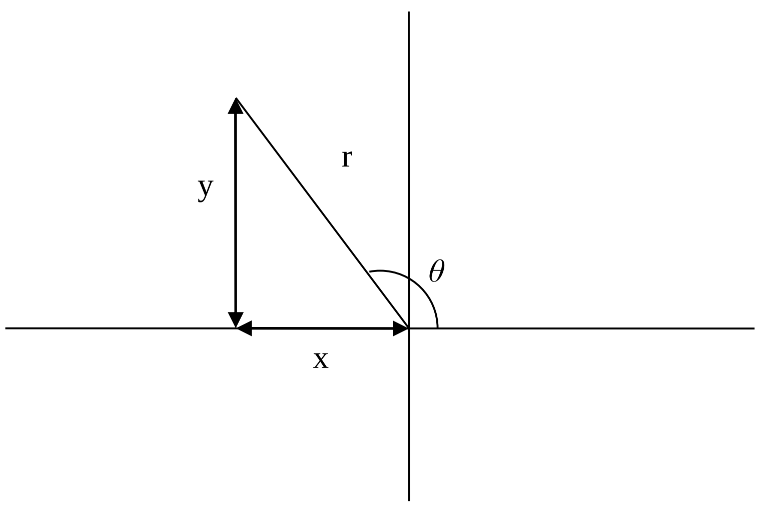
Polar coordinates: (r, θ)

1. Stand on the origin facing the positive *x*-axis.
2. Rotate counterclockwise by the angle θ.
3. Walk (or fire your cannon) a distance *r* in the direction you are facing.

Important Notes:

- If θ is *positive*, then you rotate *counterclockwise* from the positive *x*-axis.
  If θ is *negative*, then you rotate *clockwise* from the positive *x*-axis.
- If *r* is *positive* then you walk *forward* from the direction you are facing.
  If *r* is *negative* then you walk *backward* from the direction you are facing.

Note that these methods both get you to the same location.
Here is an illustration of this situation of both methods.



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You can go back and forth between Cartesian and polar coordinates, by using basic trig facts.

Namely: $\boldsymbol{x = r\cos(\theta)}$, $\boldsymbol{y = r\sin(\theta)}$, $\boldsymbol{x^2 + y^2 = r^2}$, $\boldsymbol{\tan(\theta) = \frac{y}{x}}$.

Graphing

When you first learned to graph equations such as $y = x^2$ and $y = e^x$, you had to plot lots of points to get an idea of what the graph looked like. That is, you had to make a table by selecting values of x and computing values of y. And then you plotted the (x, y) coordinates using the Cartesian coordinate method. Note that as x gets bigger the graph moves to the right as we are accustomed.

Since we are new to polar coordinates, you will have to use the same idea to plot polar equations.
You have to make a table by selecting values of θ and computing values of r. And then you plot (r, θ) using the Polar coordinate method. Here as θ increases the graph moves in a counterclockwise direction, so the graphs are often "spiraling" in some way.

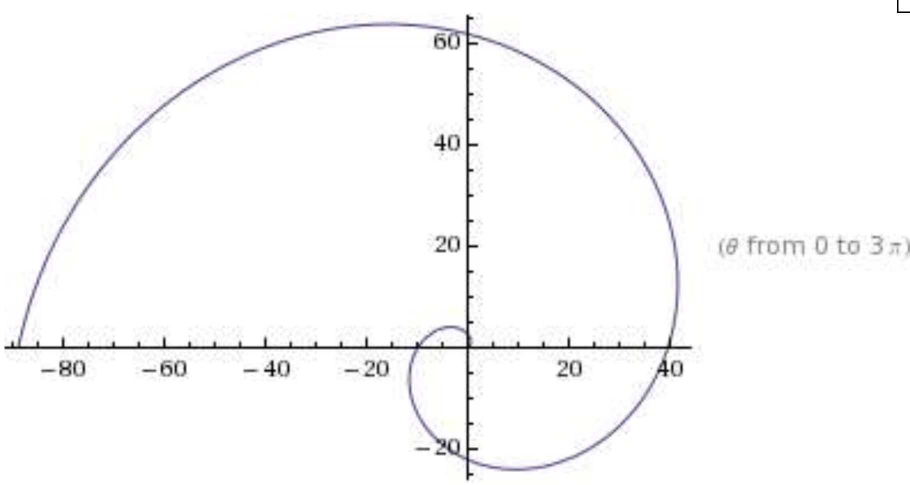
There is one other plotting option for Polar coordinates. You can first change the variables from r and θ to x and y. Then use what you know about plotting in Cartesian coordinates. In order to change the variables you will need to use the identities from and you will have to do some algebra. You will practice these ideas in the worksheet and homework.

Example 1:

$r = \theta^2$

To the right you can see that I calculated a few points. From the table we see that as θ spirals around r gets bigger and bigger. So $r = \theta^2$ is spiraling out.
Below is the picture as far as we have plotted.

θ	r
0	0
$\pi/2$	$(\pi/2)^2 \approx 2.47$
π	$(\pi)^2 \approx 9.87$
$3\pi/2$	$(3\pi/2)^2 \approx 22.21$
2π	$(2\pi)^2 \approx 39.48$
$5\pi/2$	$(5\pi/2)^2 \approx 61.69$
3π	$(3\pi)^2 \approx 88.83$



If r is any strictly increasing (or strictly decreasing) function of θ, then the graph will have this spiraling in or spiraling out behavior. In this case, r is increasing for the theta values from 0 to 3π.

In many situations where we use polar coordinates, the function for r in terms of θ involves trig functions (which oscillate and don't strictly increase or decrease).

Example 2:

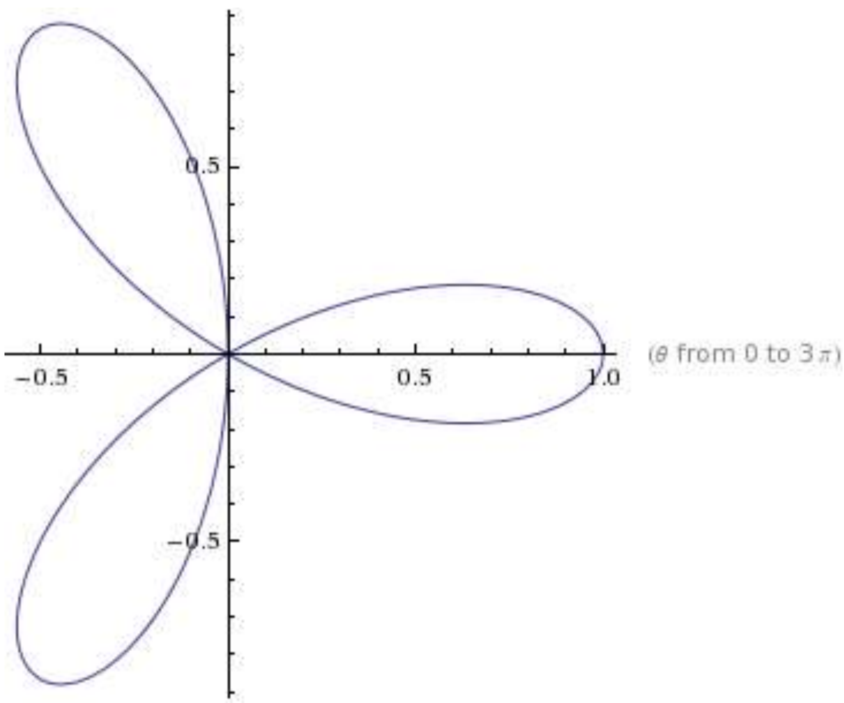
$r = \cos(3\theta)$

At the right you will see lots of points that we plotted. Note that we picked points to plot so that the input to cosine (that is, 3θ, comes out to be the multiples of $\pi/4$, which we know for cosine).

Observations:

- As θ spirals from 0 to $\pi/6$, r decreases to the origin.
- As θ spirals from $\pi/6$ to $\pi/2$, r is negative (so in the third quadrant) and gets to maximum of 1 before returning to the origin.
- As θ spirals from $\pi/2$ to $5\pi/6$, r is positive (so in the second quadrant) and gets to a maximum of 1 before returning to the origin.
- As θ spirals from $5\pi/6$ to π, r is negative (so in the fourth quadrant) and ends up back at the location we started out.
- Then it all repeats. Here is the graph:

θ	r
0	1
$\pi/12$	$\sqrt{2}/2$
$\pi/6$	0
$\pi/4$	$-\sqrt{2}/2$
$\pi/3$	-1
$5\pi/12$	$-\sqrt{2}/2$
$\pi/2$	0
$7\pi/12$	$\sqrt{2}/2$
$4\pi/3$	1
$3\pi/4$	$\sqrt{2}/2$
$5\pi/6$	0
$11\pi/12$	$-\sqrt{2}/2$
π	-1



Now look at the worksheet, lecture notes and textbook for more examples.